Integral Solutions of Ternary Quadratic Diophantine Equation $11x^2 - 3y^2 = 8z^2$

[I]R.Anbuselvi, [II]S.Jamuna Rani
[I]Associate Professor, Dept. of Mathematics, ADM College for Women, Nagapattinam, Tamilnadu, India
[II]Asst Professor, Dept. of Computer Applications, Bharathiyar College of Engineering and Technology, Karaikal, Puducherry, India

Abstract
The ternary quadratic Diophantine equation given by $11x^2 - 3y^2 = 8z^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords
Ternary Quadratic, Integral Solutions, Polygonal Numbers.

Introduction

Ternary quadratic equations are rich in variety [1-3].For an extensive review of sizable literature and various problems, one may refer [1-16]. In this communication, we consider yet another interesting ternary quadratic equation $11x^2 - 3y^2 = 8z^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special Polygonal numbers are presented.

Notations Used

- $t_{m,n}$- Polygonal number of rank 'n' with size 'm'
- P_n^m- Pyramidal number of rank 'n' with size 'm'

Methods of Analysis

The Quadratic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

$$11x^2 - 3y^2 = 8z^2 \qquad (1)$$

On substituting the linear transformations

$$x = u + v;$$
$$y = u - v \qquad (2)$$

in (1), it leads to

$$u^2 = 33v^2 + z^2 \qquad (3)$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

Pattern I

Equation (3) is satisfied by

$$\left.\begin{array}{l} u = 33m^2 + n^2 \\ v = 2mn \\ z = 33m^2 - n^2 \end{array}\right\}$$

Substituting (4)and (3)in (2), the corresponding non-zero distinct integral solution of(1)are given by

$$\left.\begin{array}{l} x(m,n) = x = 33m^2 + n^2 + 6mn \\ y(m,n) = y = 33m^2 + n^2 + 22mn \\ z(m,n) = z = 33m^2 - n^2 \end{array}\right\}$$

Properties

1. $x(m,4) - 48\, t_{3,m} - t_{18,m} \equiv 16 (mod\ 8)$
2. $y(m,2) - 66\, t_{3,m} \equiv 4\ (mod\ 11)$
3. $z(m,1) - t_{68,m} + 1 \equiv 0\ (mod\ 32)$
4. $x(1,n) - 2\, t_{3,m} \equiv 3\ (mod\ 5)$
5. $y(z,n) - 2\, t_{3,n} \equiv 3\ (mod\ 43)$
6. $y\,(a, a+1) - x(a, a+1) - 32\, t_{3,a} \equiv 0$
7. $y\big(a, a(a+1)\big) - x(a, a+1) - 32\, P_a^5 \equiv 0$
8. $y\big(a(a+1)(a+2)\big) - x\big(a(a+1)(a+2)\big) - 96P_a^3 \equiv 0$
9. $x + y + z$ is expressed as the difference of two squares when $m = n$
10. Each of the following expression represents a nasty number
    (i) $x(a,a) + y(a,a)$
    (ii) $y(a,a) - z(a,a)$

Note

Instead of (2) one may also consider the linear transformation

$$x = u - 3v\ ; \qquad y = u - 11v \qquad (6)$$

For this choice, the corresponding integer solutions to (1) are

$$x = m^2 + 33n^2 - 6mn$$
$$y = m^2 + 33n^2 - 22mn$$
$$z = m^2 - 33n^2$$

Pattern II

Equation (3) is written as

$$\frac{u-z}{3v} = \frac{11v}{u+z} = \frac{m}{n} \quad , \quad n \neq 0 \qquad (7)$$

Which is equivalent to the system of equations

$$um + zm - 11vn = 0$$
$$un - zn - 3vm = 0$$

From which we get

$$\left.\begin{aligned} u &= 3m^2 + 11\,n^2 \\ v &= 2mn \\ z &= 3m^2 - 11\,n^2 \end{aligned}\right\} \qquad (8)$$

Using (8)in (2) we obtain the integer solutions to(1) as given below

$$\left.\begin{aligned} x &= 3m^2 + 11n^2 - 6mn \\ y &= 3m^2 + 11\boldsymbol{n^2} - 22mn \\ z &= 3m^2 - 11n^2 \end{aligned}\right\} \qquad (9)$$

Properties

1. $x(a, a+1) - y(a, a+1) - 36\,t_{3,a} \equiv 0$
2. $x(a, 1) - y(a, 1) \equiv 0 (mod\ 16)$
3. $8x(a^2, a+1) - 8y(a^2, a+1) - 256P_a^5 \equiv 0$
4. $x(a, a) - z(a, a) \equiv 0$
5. $x(a, a) + y(a, a) \equiv 0$
6. $x\big(a, a(a+1)\big) - y\big(a, a(a+1)\big) - 32P_a^5 \equiv 0$
7. $x\big(a, (a+1)(a+2)\big) - y\big(a, (a+1)(a+2)\big) - 96\,P_a^3 \equiv 0$
8. $z(1, b) - y(1, b) + t_{46,b} \equiv 0$
9. Each of the following expressions represents a nasty number
   (i) $3[x(a, a) + y(a, a)]$
   (ii) $6[y(a, a) + z(a, a)]$
   (iii) $3[x(a, -a) + y(a, -a) + z(a, -a)]$
   (iv) $2[x(a, -a) - y(a, -a) - z(a, -a)]$

Pattern III

Equation (3) can be written as

$$z = u^2 - 33v^2 \qquad (10)$$

Take $z = a^2 - 33b^2$ (11)

Using (10)and (11) and equating positive and negative factors, we get

$$\left.\begin{aligned} u &= a^2 + 33b^2 \\ v &= 2ab \end{aligned}\right\} \qquad (12)$$

From (12)and (13)we get

$$\left.\begin{aligned} x &= a^2 + 33b^2 + 6ab \\ y &= a^2 + 33b^2 - 22ab \end{aligned}\right\} \qquad (13)$$

$$z = a^2 - 33\boldsymbol{b^2} \qquad (14)$$

Thus (13) and (14)represent non-zero distinct integer solutions to (1)

Properties

1. $x(a, 1) - 2t_{3,a} \equiv 33 (mod\ 5)$
2. $y(a, 1) - 2t_{3,a} \equiv 33 (mod\ 21)$
3. $y(a, b) - x(a, b) = 16ab$
4. $y(a, a+1) - x(a, a+1) - 32t_{3,a} \equiv 0$
5. $y(a, a(a+1)) - x(a, (a+1)) - 32\,P_a^5 \equiv 0$
6. $y\big(a, (a+1)(a+2)\big) - x\big(a, (a+1)(a+2)\big) - 96\,P_a^3 \equiv 0$
7. $y(a, 1) + z(a, 1) - t_{98,a} + t_{94,a} \equiv 0\ (mod\ 4)$
8. Each of the following expression represents a nasty number
   (i) $x(b, -b) - z(b, -b)$
   (ii) $8[y(2a, a) + z(2a, a) - x(2a, a)]$

Pattern IV

Write (3) as

$$z^2 + 33v^2 = u^2 \times 1 \qquad (15)$$

Assume $u = a^2 + 33\mathrm{b}^2$ (16)

$$1 = \frac{(4 + i\sqrt{33})(4 - i\sqrt{33})}{49} \qquad (17)$$

Using (16) and (17) in (15)and employing the method of factorization, define

$$z + i\sqrt{33}\ v = \left(a + i\sqrt{33}b\right)^2 (45 + i\sqrt{33}\,)$$

Equating real and imaginary part, we get

$$\left.\begin{aligned} z &= \frac{1}{7}(f(a, b)) \\ &= \frac{1}{7}(4a^2 - 66ab - 132b^2) \\ v &= \frac{1}{7}(g(a, b)) \\ &= \frac{1}{7}(a^2 - 33b^2 + 8ab) \end{aligned}\right\} \qquad (18)$$

ISSN : 2394-2975 (Online)
ISSN : 2394-6814 (Print)

As our interest is on finding integer solutions, it is seen that the values of x, y and z are integers when both a and b are of the same parity. Thus by taking $a = 7a$, $b = 7b$ in (18) and substituting the corresponding values of u,v in (2) the non-zero integral solution of (1) are given by

$$\left.\begin{aligned} x &= 70m^2 + 924n^2 + 168mn \\ y &= 126m^2 - 924n^2 + 616mn \\ z &= 28m^2 - 924n^2 - 462mn \end{aligned}\right\} \qquad (19)$$

Properties

1. $x(a(a+1)), y(a, a+1)) - 980\ Obl_a \equiv 0 \ (mod\ 14)$
2. $x(a, a(a+1)), y(a, a(a+1)) = 1568P_a^5 - 196_a^2$
3. $x(a,1) + y(a,1) - z(a,1)196t_3^a \equiv 0 \ mod\ (14)$
4. $x(2,b) + y(2,b) = 784\ mod(1568)$
5. $x(a,1) + y(a,1) - z(a,1) - 196_3^a \equiv 0\ (mod\ 980)$

Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

References

[1]. GopalanMA,Manjusomanath,VanithaN.On Ternary Cubic Equation $x^2 - y^2 = z^3$, ActaCiencia Indica,2007, vol XXXIIIM, No-3,705-707.

[2]. GopalanMA,SangeetheG.On the Ternary Cubic Diophantine Equation $y^2 = Dx^2 + z^3$ Archimedes J.Math, 2011,1(1):7-14.

[3]. GopalanMA,VijayashankarA,VidhyalakshmiS.Integral solutions of Ternary cubic Equation,, Archimedes J.Math,2011;1(1):59-65.

[4]. GopalanM.A,VidhyalakshmiS,KavithaA,Observation on the Ternary Cubic Equation $x^2 + y^2 + xy = 12z^3$ Antarctica J.Math,2013;10(5):453-460.

[5]. GopalanM.A,GeethaD,Lattice points on the Hyperboloid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$ Impact J.Sci.Tech,2010,4,23-32.

[6]. GopalanM.A,VidhyalakshmiS,KavithaA,Integral points on the Homogenous Cone $z^2 = 2x^2 - 7y^2$,The Diophantus J.Math,2012,1(2) 127-136.

[7]. GopalanM.A,VidhyalakshmiS,SumathiG,Lattice points on the Hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$, The Diophantus J.Math,2012,1(2),109-115.

[8]. GopalanM.A,VidhyalakshmiS,LakshmiK,Integral points on the Hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$, Diophantus J.Math,2012,1(2),99-107.

[9]. GopalanM.A,VidhyalakshmiS,MallikaS,Observation on Hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$ Bessel J.Math,2012,2(3),221-226.

[10]. GopalanM.A,VidhyalakshmiS,Usha Rani T.R,MallikaS,Integral points on the Homogenous cone $6z^2 + 3y^2 - 2x^2 = 0$ Impact J.Sci.Tech,2012,6(1),7-13.

[11]. GopalanM.A,VidhyalakshmiS,LakshmiK,Lattice points on the Elliptic Paraboloid, $16y^2 + 9z^2 = 4x^2$ Bessel J.Math,2013,3(2),137-145.

[12]. GopalanM.A,VidhyalakshmiS,UmaraniJ,Integral points on the Homogenous Cone $x^2 + 4y^2 = 37z^2$,Cayley J.Math,2013,2(2),101-107.

[13]. MeenaK,VidhyalakshmiS,GopalanM.A,PriyaK,Integral points on the cone $3(x^2 + y^2) - 5xy = 47z^2$, Bulletin of Mathematics and Statistics and Research,2014,2(1),65-70.

[14]. GopalanM.A,VidhyalakshmiS,NivethaS,on Ternary Quadratic Equation $4(x^2 + y^2) - 7xy = 31z^2$ Diophantus J.Math,2014,3(1),1-7.

[15]. GopalanM.A,VidhyalakshmiS,ShanthiJ,Lattice points on the Homogenous Cone $8(x^2 + y^2) - 15xy = 56z^2$ Sch Journal of Phy Math Stat,2014,1(1),29-32.

[16]. MeenaK,GopalanM.A,VidhyalakshmiS,ManjulaS,Thirunir aiselvi, N,On the Ternary quadratic Diophantine Equation $8(x^2 + y^2) + 8(x + y) + 4 = 25z^2$,International Journal of Applied Research,2015,1(3),11-14.

Reference Books

[1]. Dickson IE, Theory of Numbers, vol2. Diophantine analysis, New York,Dover,2005

[2]. MordellLJ.Diophantine Equations Academic Press,New York,1969

[3]. Carmichael RD, The Theory of numbers and Diophantine Analysis, NewYork,Dover,1959.