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Fixed Points of n-periodic and uniformly *p*-lipschtizian mappings in Hilbert Spaces

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Abstract
Many fixed point theorems in metric spaces satisfying contraction-like conditions can be ex- tended to Banach spaces and Locally convex spaces. The main purpose of this paper is to study the Lipschitz constants of n-periodic mappings in Hilbert spaces that guarantee the existence of fixed points and retractions on the fixed point set. In addition, we give new estimates for the n-periodic and uniformly p-lipschitzian mappings in Hilbert spaces that guarantee the existence of fixed points.

Keywords
Fixed point, periodic mapping, Hilbert space, k-lipschitzian mapping, uniformly p-lipschitzian mapping and parallelogram law.

1 Introduction

Let C be a nonempty closed convex subset of a Banach space X.

Definition 1.1. (Fixed point) The fixed point set of $T : C \to C$ is denoted by

$$\text{Fix } T = \{p \in C : Tp = p\}.$$

Definition 1.2. (k-lipschitzian mapping) A mapping $T : C \to C$ is called k-lipschitzian if for all $x, y \in C$, $||Tx - Ty|| \le k||x - y||$. We write $T \in \alpha(k)$ and if k (Lipschitz constant) is the smallest number such that $T \in \alpha(k)$, then we write $T \in \alpha_0(k)$.

Definition 1.3. (Periodic map) A mapping $T : C \to C$ is said to be periodic if there exists an integer $n \ge 2$ such that $T^n = I$, where I is an identity map. If $n = 2$, then T is called involution.

Definition 1.4. (uniformly p-lipschitzian mapping) A mapping $T : C \to C$ is said to be uniformly p-lipschitzian if there is $p > 0$ such that for $i \in \mathbb{N}$ and $x, y \in C$,

$$||T^i x - T^i y|| \le p||x - y||.$$

We say $T \in \mathcal{U}(p)$. If $p = \min\{l_1 : ||T^i x - T^i y|| \le l_1||x - y||, i \in \mathbb{N}, x, y \in C\}$, then we write $T \in \mathcal{U}_0(p)$. However, there are cases such that T is n-periodic, $T \in \alpha_0(k)$ and $T \in \mathcal{U}_0(p)$ with $p < k^{n-1}$.

Definition 1.5. ((n, a)-rotative k-lipschitzian mapping) Denote $\Phi(n, a, k, C) = \Phi(n, a, C) \cap \alpha(k)$. A mapping $T \in \Phi(n, a, k, C)$ is said to be (n, a)-rotative k-lipschitzian on C.

Remark 1.6. For fixed $n \in \mathbb{N}$ define

$$\gamma_n(a) = \inf\{k > 1 | T : C \to C \text{ such that } T \in \Phi(n, a, k, C) \text{ and Fix } T = \emptyset\},$$

where Fix T denotes the set of all fixed points of T. The definition of $\gamma_n(a)$ implies that for an arbitrary set C, if $T \in \Phi(n, a, k, C)$ and $k < \gamma_n(a)$, then T has at least one fixed point. In general, $\gamma_n(a)$ are unknown.

In 1970, Goebel [1] showed that involutions have a fixed point if they are k-lipschitzian for $k < 2$ in a Banach space and for $k < \sqrt{5}$ in Hilbert space, where k is the Lipschitz constant.

In 1986, Koter [2] proved that $\gamma_2^{\mathcal{H}} \geq \sqrt{\pi^2 - 3} \approx 2.6209$ for Hilbert space $\mathcal{H}$.

In 2000, Koter-Môrgowska [3] gave the following estimations $\gamma_3^{\mathcal{H}} \geq 1.3666$, $\gamma_4^{\mathcal{H}} \geq 1.1962, \gamma_5^{\mathcal{H}} \geq 1.0849$ and $\gamma_6^{\mathcal{H}} \geq 1.0228$ for n-periodic and k-lipschitzian mappings in Hilbert space. It was also proved that for uniformly k-lipschitzian for $k > 1$, $\tilde{\gamma}_3^{\mathcal{H}} \geq 1.5447, \tilde{\gamma}_4^{\mathcal{H}} \geq 1.2418, \tilde{\gamma}_5^{\mathcal{H}} \geq 1.1429$ and $\tilde{\gamma}_6^{\mathcal{H}} \geq 1.0277$.

2 Preliminaries

In this section we recall some basic notions in functional analysis and provide brief introduction to Hilbert spaces.

Lemma 2.1. ([4]) Let X be a complete metric space and $T : X \to X$ a continuous mapping. Suppose there are $u : X \to X, 0 < A < 1$ and $B > 0$ such that for every $x \in X$:

(i) $d(Tu(x), u(x)) \leq Ad(Tx, x)$,

(ii) $d(u(x), x) \leq Bd(Tx, x)$.

Then Fix $T \neq \emptyset$. If we define $R(x) = \lim\limits_{n\to\infty} u^n(x)$ and u is a continuous mapping, then R is a retraction from X to Fix T. Suppose $T \in \alpha(k)$:

(a) If $k < 1$, then T has a unique fixed point.

(b) If $k = 1$, then R is a nonexpansive mapping.

(c) If $k > 1$ and $D = \operatorname{diam}(X) = \sup\{||u-v|| : u, v \in X\} < \infty$, then R is a Hölder continuous retraction from X to Fix T.

Lemma 2.2. ([5]) Let $T : C \to C$ be k-lipschitzian mapping. Let $A, B \in \mathbb{R}$ with $0 \leq A < 1$ and $B > 0$. If for arbitrary $x \in C$ there exists $z \in C$ such that

$$||Tz - z|| \leq A||Tx - x||$$

and

$$||z - x|| \leq B||Tx - x||,$$

then T has a fixed point in C.

Lemma 2.3. Let $\mathcal{H}$ be a real Hilbert space. If $w, v \in \mathcal{H}$ and $\alpha \in [0, 1]$, then

$$||(1-\alpha)w + \alpha v||^2 = (1-\alpha)||w||^2 + \alpha||v||^2 - \alpha(1-\alpha)||w - v||^2. \tag{1}$$

Proof. Since $w, v \in \mathcal{H}$ and $\alpha \in [0, 1]$, then

$$\begin{aligned}
||(1-\alpha)w + \alpha v||^2 &= \langle (1-\alpha)w + \alpha v, (1-\alpha)w + \alpha v\rangle \\
&= \langle (1-\alpha)w, (1-\alpha)w\rangle + \langle (1-\alpha)w, \alpha v\rangle + \langle \alpha v, (1-\alpha)w\rangle + \langle \alpha v, \alpha v\rangle \\
&= (1-\alpha)^2||w||^2 + \alpha(1-\alpha)\langle w, v\rangle + \alpha(1-\alpha)\langle v, w\rangle + \alpha^2||v||^2 \\
&= (1-\alpha)^2||w||^2 + \alpha(1-\alpha)\langle w, v\rangle + \alpha(1-\alpha)\langle w, v\rangle + \alpha^2||v||^2 \\
&= (1-\alpha)^2||w||^2 + 2\alpha(1-\alpha)\langle w, v\rangle + \alpha^2||v||^2. \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
\alpha(1-\alpha)||w-v||^2 &= \alpha(1-\alpha)\langle w-v, w-v\rangle \\
&= \alpha(1-\alpha)[\langle w,w\rangle - \langle w,v\rangle - \langle v,w\rangle + \langle v,v\rangle] \\
&= \alpha(1-\alpha)[\|w\|^2 - 2\langle w,v\rangle + \|v\|^2] \\
&= \alpha(1-\alpha)\|w\|^2 - 2\alpha(1-\alpha)\langle w,v\rangle + \alpha(1-\alpha)\|v\|^2. \qquad (3)
\end{aligned}$$

It immediately follows from Equation (2) and (3) that

$$||(1-\alpha)w + \alpha v||^2 = (1-\alpha)||w||^2 + \alpha||v||^2 - \alpha(1-\alpha)||w-v||^2,$$

as desired. □

Lemma 2.4. (Generalization of parallelogram law) Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$ and $b_i \in [0,1]$ for $i = 1, 2, ..., n$, such that $\sum_{i=1}^{n} b_i = 1$. If $x_i \in \mathcal{H}$ for $i = 1, 2, ..., n$, then

$$||\sum_{i=1}^{n} b_i x_i||^2 = \sum_{i=1}^{n} b_i||x_i||^2 - \sum_{1\le i<j\le n} b_i b_j||x_i - x_j||^2. \qquad (4)$$

Proof. (Induction) Denote Equation (4) by $R(n)$. Consider $R(2)$ and using Equation (1), we obtain

$$\begin{aligned}
\|b_1x_1 + b_2x_2\|^2 &= b_1\|x_1\|^2 + b_2\|x_2\|^2 - b_1b_2\|x_1 - x_2\|^2 \\
&= \sum_{i=1}^{2} b_i\|x_i\|^2 - \sum_{1\le i<j\le 2} b_ib_j||x_i - x_j||^2.
\end{aligned}$$

Hence $R(2)$ is true. Suppose $R(k)$ is true for some $k < n \in \mathbb{N}$, then

$$||\sum_{i=1}^{k} b_i x_i||^2 = \sum_{i=1}^{k} b_i||x_i||^2 - \sum_{1\le i<j\le k} b_i b_j||x_i - x_j||^2.$$

Considering $R(k+1)$ we obtain

$$\begin{aligned}
\|\sum_{i=1}^{k+1} b_i x_i\|^2 &= \|\sum_{i=1}^{k} b_i x_i + b_{k+1}x_{k+1}\|^2 \\
&= \|\sum_{i=1}^{k} b_i x_i\|^2 + b_{k+1}\|x_{k+1}\|^2 - b_{k+1}\|\sum_{i=1}^{k} b_i x_i - x_{k+1}\|^2 \\
&= \|\sum_{i=1}^{k} b_i x_i\|^2 + b_{k+1}\|x_{k+1}\|^2 - b_{k+1}\|\sum_{i=1}^{k} b_i(x_i - x_{k+1})\|^2 \\
&= \sum_{i=1}^{k} b_i||x_i||^2 - \sum_{1\le i<j\le k} b_ib_j||x_i - x_j||^2 + b_{k+1}\|x_{k+1}\|^2 - b_{k+1}\sum_{i=1}^{k} b_i\|x_i - x_{k+1}\|^2 \\
&= \sum_{i=1}^{k} b_i||x_i||^2 + b_{k+1}\|x_{k+1}\|^2 - \sum_{1\le i<j\le k} b_ib_j||x_i - x_j||^2 - \sum_{i=1}^{k} b_ib_{k+1}\|x_i - x_{k+1}\|^2 \\
&= \sum_{i=1}^{k+1} b_i||x_i||^2 - \sum_{1\le i<j\le k+1} b_ib_j||x_i - x_j||^2.
\end{aligned}$$

Thus $R(n)$ is true for all $n \in \mathbb{N}$. □

Lemma 2.5. ([4]) Let $n \in \mathbb{N}$ and $T : \mathbb{X} \to \mathbb{X}$ be n-periodic and k-lipschitzian mapping, where $\mathbb{X}$ is a nonempty closed convex subset of a Hilbert space. Let $b_i > 0$ for $i = 1, 2, ..., n$, such that $\sum_{i=1}^{n} b_i = 1$ and $b_0 = b_n$. If for $x \in \mathbb{X}$ and

$$z = \sum_{i=1}^{n} b_i T^i x,$$

then

$$||z - x|| \leq \sum_{i=1}^{n-1} b_i ||T^i x - x|| \tag{5}$$

and

$$||z - Tz||^2 \leq \sum_{0 \leq j < i \leq n-1} F(k, b_j, b_{j+1}, b_i, b_{i+1}) ||T^j x - T^i x||^2, \tag{6}$$

where $F(k, x, y, u, w) = k^2(yu + xw - xu) - xu$.

3 Main Results

3.1 Estimations of $\gamma_n^{\mathcal{H}}$ for $T \in \alpha(k)$

In this section, we want to estimate the value of $\gamma_n^{\mathcal{H}}$ for k-lipschitzian and n-periodic mapping using $b_i = \frac{1}{n}$. Simulations will be done using Sage and Octave.

Theorem 3.1.1. Let $\mathbb{X}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $T : \mathbb{X} \to \mathbb{X},\ T \in \alpha(k)$ be an n-periodic mapping with $n \geq 3$. If $x \in \mathbb{X}$ and

$$z = \frac{1}{n} \sum_{i=1}^{n} T^i x$$

then

$$\begin{aligned} ||z - Tz||^2 &\leq \frac{1}{n^2} \left[(k^2 - 1) k^{2(n-1)} + \sum_{j=2}^{n-1} (k^{2j} - 1) \left(\frac{k^{n-j} - 1}{k - 1} \right)^2 \right] ||x - Tx||^2 \\ &= A(k) ||x - Tx||^2. \end{aligned}$$

Thus, if $A(k) < 1$, then Fix $T \neq \emptyset$ and Fix T is a retract of $\mathbb{X}$. If $k = 1$, Fix T is a nonexpansive retract of $\mathbb{X}$ and if $k > 1$ with $\mathbb{X}$ bounded, then Fix T is a Hölder continuous retract of $\mathbb{X}$.

Proof. If $b_i = \frac{1}{n}$ in Lemma 2.5, then

$$\begin{aligned} F &= k^2 (b_{j+1} b_i + b_j b_{i+1} - b_j b_i) - b_j b_i \\ &= k^2 \left(\frac{1}{n^2} + \frac{1}{n^2} - \frac{1}{n^2} \right) - \frac{1}{n^2} \\ &= \frac{k^2 - 1}{n^2}. \end{aligned}$$

It follows from Equation (6) that

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$$||z - Tz||^2 \leq \sum_{0\leq j<i\leq n-1} \frac{k^2-1}{n^2}||T^jx - T^ix||^2. \tag{7}$$

If $j < i$, then

$$||T^jx - T^ix|| \leq k^j||x - T^{i-j}x||. \tag{8}$$

If $j < n-1$, then

$$\begin{aligned} ||x - T^jx|| &= ||x - Tx + Tx - T^2x + T^2x - ... + T^{j-1}x - T^jx|| \\ &\leq ||x - Tx|| + ||Tx - T^2x|| + ... + ||T^{j-1}x - T^jx|| \\ &= \sum_{i=0}^{j-1} ||T^ix - T^{i+1}x|| \\ &\leq \sum_{i=0}^{j-1} k^i||x - Tx|| \\ &= \frac{k^j-1}{k-1}||x - Tx||. \end{aligned} \tag{9}$$

Since T is n-periodic, we obtain

$$||x - T^{n-1}x|| = ||T^nx - T^{n-1}x|| \leq k^{n-1}||Tx - x||. \tag{10}$$

It follows from Equation (7) that

$$\begin{aligned} ||z - Tz||^2 &\leq \frac{k^2-1}{n^2}\sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1} ||T^jx - T^ix||^2 \\ &= \frac{k^2-1}{n^2}\sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1} ||T^j(x - T^{i-j}x)||^2 \\ &\leq \frac{k^2-1}{n^2}\sum_{j=0}^{n-2}k^{2j}\sum_{i=j+1}^{n-1} ||x - T^{i-j}x||^2 \\ &= \frac{k^2-1}{n^2}\sum_{j=0}^{n-2}k^{2j}\sum_{i=1}^{n-j-1} ||x - T^ix||^2 \\ &= \frac{k^2-1}{n^2}\sum_{i=1}^{n-1}\left(\sum_{j=0}^{n-i-1} k^{2j}\right) ||x - T^ix||^2 \\ &= \sum_{i=1}^{n-1}\frac{k^{2(n-i)}-1}{n^2}||x - T^ix||^2 \\ &= \frac{1}{n^2}\sum_{j=1}^{n-1}(k^{2(n-j)}-1)||x - T^jx||^2 \\ &= \frac{1}{n^2}\sum_{j=1}^{n-2}(k^{2(n-j)}-1)||x - T^jx||^2 + \frac{1}{n^2}(k^2-1)||x - T^{n-1}x||^2. \end{aligned}$$

Using Equation (9) and (10), we obtain

$$\begin{aligned}
&\leq \frac{1}{n^2}\sum_{j=1}^{n-2}(k^{2(n-j)}-1)\left(\frac{k^j-1}{k-1}\right)^2||x-Tx||^2+\frac{1}{n^2}(k^2-1)(k^{n-1})^2||x-Tx||^2\\
&= \frac{1}{n^2}\left[(k^2-1)k^{2(n-1)}+\sum_{j=2}^{n-1}(k^{2j}-1)\left(\frac{k^{n-j}-1}{k-1}\right)^2\right]||x-Tx||^2\\
||z-Tz||^2 &\leq A(k)||x-Tx||^2,
\end{aligned}$$

where

$$A(k)=\frac{1}{n^2}\left[(k^2-1)k^{2(n-1)}+\sum_{j=2}^{n-1}(k^{2j}-1)\left(\frac{k^{n-j}-1}{k-1}\right)^2\right]. \tag{11}$$

Also, from Equation (5) and (9), we have

$$\begin{aligned}
||z-x|| &\leq \frac{1}{n}\sum_{i=1}^{n-1}||x-T^ix||\\
&\leq \frac{1}{n}\sum_{i=1}^{n-1}\frac{k^i-1}{k-1}||x-Tx||\\
&= B(k)||x-Tx||,
\end{aligned}$$

where

$$0<B(k)=\frac{1}{n}\sum_{i=1}^{n-1}\frac{k^i-1}{k-1}.$$

By Lemma 2.1 , if $A(k)<1$, then Fix $T\neq\emptyset$ and Fix T is a retract of $\mathbb{X}$. If $k=1$, Fix T is a nonexpansive retract of $\mathbb{X}$ and if $k>1$ with $\mathbb{X}$ bounded, then Fix T is a Hölder continuous retract of $\mathbb{X}$. Since for fixed n, $\lim_{k\to 1}A(k)=0$, there is $k>1$ such that $A(k)<1$. This is the proof that for Hilbert space $\mathcal{H}, \gamma_n^{\mathcal{H}}>1$. From Equation (11), if $n=3$, then k satisfies the inequality

$$\begin{aligned}
\frac{1}{9}((k^2-1)k^4+(k^4-1))&<1\\
\frac{1}{9}(k^6-1)-1&<0\\
k^6-10&<0\\
k<\sqrt[6]{10}&\approx 1.4678.
\end{aligned}$$

Thus $\gamma_3^{\mathcal{H}}\geq 1.4678$. To evaluate $\gamma_4^{\mathcal{H}}$, we only need to solve $A(k)<1$ for $n=4$ as follows:

$$\begin{aligned}
\frac{1}{16}\left[(k^2-1)k^6+\sum_{j=2}^{3}(k^{2j}-1)\left(\frac{k^{4-j}-1}{k-1}\right)^2\right]&<1\\
k^8-k^6+(k^4-1)(k+1)^2+k^6-1&<16\\
k^8+k^6+2k^5+k^4-k^2-2k-17&<0. \qquad (12)
\end{aligned}$$

Using Octave to solve Equation (12), we obtain the optimal value $\gamma_4^{\mathcal{H}}\geq 1.2905$.

For $n \geq 5,\ j = 1$ and $i = n-1$, the estimate in Equation (7) improves if we take

$$||Tx - T^{n-1}x|| \leq ||x - Tx|| + ||x - T^{n-1}x|| \leq (1 + k^{n-1})||x - Tx||. \tag{13}$$

Using Equation (9) and (13), we obtain the following

$$\begin{aligned}
||z - Tz||^2 &\leq \sum_{0\leq j<i\leq 4} \frac{k^2-1}{n^2}||T^jx - T^ix||^2 \\
&= \frac{k^2-1}{25}\Big[||x - Tx||^2 + ||x - T^2x||^2 + ||x - T^3x||^2 + ||x - T^4x||^2 \\
&\quad + ||Tx - T^2x||^2 + ||Tx - T^3x||^2 + ||Tx - T^4x||^2 + ||T^2x - T^3x||^2 \\
&\quad + ||T^2x - T^4x||^2 + ||T^3x - T^4x||^2\Big] \\
&\leq \frac{k^2-1}{25}\Big[||x - Tx||^2 + (1+k)^2||x - Tx||^2 + (1+k+k^2)^2||x - Tx||^2 \\
&\quad + k^8||x - Tx||^2 + k^2||x - Tx||^2 + k^2(1+k)^2||x - Tx||^2 + (1+k^4)^2||x - Tx||^2 \\
&\quad + k^4||x - Tx||^2 + k^4(1+k)^2||x - Tx||^2 + k^6||x - Tx||^2\Big] \\
&= \frac{k^2-1}{25}\Big[1 + (1+k)^2 + (1+k+k^2)^2 + k^8 + k^2 + k^2(1+k)^2 + (1+k^4)^2 + k^4 \\
&\quad + k^4(1+k)^2 + k^6\Big]||x - Tx||^2 \\
&= \frac{2k^{10} + 2k^7 + 4k^6 + 2k^5 - 2k^2 - 4k - 4}{25}||x - Tx||^2 \\
&= A_1(k)||x - Tx||^2,
\end{aligned}$$

where

$$\begin{aligned}
A_1(k) = \frac{2k^{10} + 2k^7 + 4k^6 + 2k^5 - 2k^2 - 4k - 4}{25} < 1 \\
2k^{10} + 2k^7 + 4k^6 + 2k^5 - 2k^2 - 4k - 29 < 0.
\end{aligned} \tag{14}$$

Using Octave to solve Equation (14), we obtain $\gamma_5^{\mathcal{H}} \geq 1.2010$.

For $n \geq 6$, we shall take the following estimations: if $j = 0, i = n-2$,

$$\begin{aligned}
||x - T^{n-2}x|| &\leq ||x - T^{n-1}x|| + ||T^{n-1}x - T^{n-2}x|| \\
&\leq (k^{n-2} + k^{n-1})||x - Tx||,
\end{aligned} \tag{15}$$

and if $j = 2, i = n-1$,

$$||T^2x - T^{n-1}x|| \leq ||x - T^2x|| + ||x - T^{n-1}x|| \leq (1 + k + k^{n-1})||x - Tx||. \tag{16}$$

Using Equation (9),(15) and (16), we obtain

$$\begin{aligned}
||z - Tz||^2 &\leq \sum_{0\leq j<i\leq 5} \frac{k^2-1}{36}||T^jx - T^ix||^2 \\
&= \frac{k^2-1}{36}\Big[||x - Tx||^2 + ||x - T^2x||^2 + ||x - T^3x||^2 + ||x - T^4x||^2
\end{aligned}$$

$$
\begin{aligned}
&+||x-T^5x||^2+||Tx-T^2x||^2+||Tx-T^3x||^2+||Tx-T^4x||^2\\
&+||Tx-T^5x||^2+||T^2x-T^3x||^2+||T^2x-T^4x||^2+||T^2x-T^5x||^2\\
&+||T^3x-T^4x||^2+||T^3x-T^5x||^2\Big]\\
\leq\ & \frac{k^2-1}{36}\Big[||x-Tx||^2+(1+k)^2||x-Tx||^2+(1+k+k^2)^2||x-Tx||^2+\\
&(k^4+k^5)^2||x-Tx||^2+k^{10}||x-Tx||^2+k^2||x-Tx||^2+k^2(1+k)^2||x-Tx||^2\\
&+k^2(1+k+k^2)^2||x-Tx||^2+(1+k^5)^2||x-Tx||^2+k^4||x-Tx||^2\\
&+k^4(1+k)^2||x-Tx||^2+(1+k+k^5)^2||x-Tx||^2+k^6||x-Tx||^2\\
&+k^6(1+k)^2||x-Tx||^2\Big]\\
=\ & \frac{k^2-1}{36}(4k^{10}+2k^9+2k^8+2k^7+6k^6+8k^5+7k^4+6k^3+8k^2+6k+5)||x-Tx||^2\\
=\ & A_2(k)||x-Tx||^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&A_2(k)=\frac{k^2-1}{36}(4k^{10}+2k^9+2k^8+2k^7+6k^6+8k^5+7k^4+6k^3+8k^2+6k+5)<1\\
&4k^{12}+2k^{11}-2k^{10}+4k^8+6k^7+k^6-2k^5-3k^2-6k-41<0. \qquad (17)
\end{aligned}
$$

Using Octave to solve Equation (17), we obtain $\gamma_6^{\mathcal{H}} \geq 1.1524$. □

Corollary 3.1.2. Let $\mathcal{H}$ be a real Hilbert space and $b_i = \frac{1}{n}$, then $\gamma_3^{\mathcal{H}} \geq 1.4678$, $\gamma_4^{\mathcal{H}} \geq 1.2905$, $\gamma_5^{\mathcal{H}} \geq 1.2010$ and $\gamma_6^{\mathcal{H}} \geq 1.1524$.

3.2 Estimations of $\tilde{\gamma}_n^{\mathcal{H}}$ for $T \in \mathcal{U}(p)$ with $p < k^{n-1}$

In this section, we estimate the corresponding value of $\tilde{\gamma}_n^{\mathcal{H}}$. In Theorem 3.1.1, $T^i \in \alpha(k^i)$ was used in order to obtain the best estimation for $\gamma_n^{\mathcal{H}}$. There are n-periodic functions for each $i = 1, 2, ..., n-1$ such that $T^i \in \alpha_0(k^i)$. That is, $T^i \notin \alpha(p)$ for $p < k^i$.

Let $X = c_0^n(\mathbb{R}), C = \{(x_1, ..., x_n) \in X : x_i \geq 0, i = 1, ..., n\}$ and $k > 1$. We define a mapping $T : C \to C$ as follows:

$$T(x_1, ..., x_n) = \left(x_2, ..., x_{n-1}, kx_n, \frac{x_1}{k}\right). \qquad (18)$$

If T is n-periodic, then for each $i = 1, 2, .., n-1$, we have $T^i \in \alpha(k)$.

Let X be a Banach space and C a nonempty closed convex subset of X, we define

$$\tilde{\gamma}_n^X = \inf\{p : \exists (C \subset X, T : C \to C), T^n = I, T \in \mathcal{U}_0(p), \text{Fix } T = \emptyset\}. \qquad (19)$$

Since $T \in \alpha_0(k)$ implies $T \in \mathcal{U}_0(p)$ with $p \geq k$, then it is clear that $\tilde{\gamma}_n^X \geq \gamma_n^X$.

Theorem 3.2.1. Let $\mathbb{X}$ be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$ and $T : \mathbb{X} \to \mathbb{X}$, $T \in \mathcal{U}(p)$ be an n-periodic mapping with $n \geq 3$. If $x \in \mathbb{X}$ and

$$z = \frac{1}{n}\sum_{i=1}^{n} T^i x$$

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then

$$\begin{aligned} ||z-Tz||^2 &\leq \left\{\frac{1}{n^2}\left[\sum_{i=1}^{n-2}((n-i-1)p^4-(n-i-2)p^2-1)\right](1+(i-1)p)^2+(p^2-1)p^2\right\}||x-Tx||^2 \\ &= C(p)||x-Tx||^2. \end{aligned}$$

Thus, if $C(p)<1$, then Fix $T\neq\emptyset$.

Proof. If $i<n-1$, then

$$\begin{aligned} ||x-T^ix|| &= ||x-Tx+Tx-T^2x+T^2x-...+T^{j-1}x-T^jx|| \\ &\leq ||x-Tx||+||Tx-T^2x||+...+||T^{j-1}x-T^jx|| \\ &\leq \sum_{i=1}^{j-1}||T^ix-T^{i+1}x||+||x-Tx|| \\ &\leq \sum_{i=1}^{j-1}p||x-Tx||+||x-Tx|| \\ &= [1+(i-1)p]||x-Tx||. \end{aligned} \tag{20}$$

Since T is n-periodic, we obtain

$$||x-T^{n-1}x|| = ||T^nx-T^{n-1}x|| \leq p||Tx-x||. \tag{21}$$

Using Theorem 3.1.1, Equation (20) and (21), we obtain the following

$$\begin{aligned} ||z-Tz||^2 &\leq \sum_{0\leq j<i\leq n-1}\frac{p^2-1}{n^2}||T^jx-T^ix||^2 \\ &= \frac{p^2-1}{n^2}\sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1}||T^jx-T^ix||^2 \\ &= \frac{p^2-1}{n^2}\sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1}||T^jx-T^ix||^2+\frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \\ &\leq \frac{p^2-1}{n^2}p^2\sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1}||x-T^{i-j}x||^2+\frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \\ &= \frac{p^2(p^2-1)}{n^2}\left[\sum_{i=2}^{n-1}||x-T^{i-1}x||^2+\sum_{i=3}^{n-1}||x-T^{i-2}x||^2+\sum_{i=4}^{n-1}||x-T^{i-3}x||^2+...\right. \\ &\quad \left.+\sum_{i=n-1}^{n-1}||x-T^{i-n+2}x||^2\right]+\frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \\ &= \frac{p^2(p^2-1)}{n^2}\Big[(||x-Tx||^2+||x-T^2x||+...+||x-T^{n-2}x||^2) \\ &\quad (||x-Tx||^2+||x-T^2x||+...+||x-T^{n-3}x||^2) \\ &\quad (||x-Tx||^2+||x-T^2x||+...+||x-T^{n-4}x||^2) \\ &\quad +...+||x-T^{n-1}x||^2\Big]+\frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \\ &= \frac{p^2(p^2-1)}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-i-1}||x-T^ix||^2+\frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{p^2(p^2-1)}{n^2}\sum_{i=1}^{n-1}(n-i-1)||x-T^ix||^2 + \frac{p^2-1}{n^2}\sum_{i=1}^{n-1}||x-T^ix||^2 \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n-1}((n-i-1)p^4-(n-i-2)p^2-1)\right]||x-T^ix||^2 \qquad (22)\\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n-2}((n-i-1)p^4-(n-i-2)p^2-1)\right]||x-T^ix||^2 + (p^2-1)||x-T^{n-1}x||^2 \\
&\le \left\{\frac{1}{n^2}\left[\sum_{i=1}^{n-2}((n-i-1)p^4-(n-i-2)p^2-1)\right](1+(i-1)p)^2 \right.\\
&\quad \left. +(p^2-1)p^2\right\}||x-Tx||^2 \\
&= C(p)||x-Tx||^2,
\end{aligned}
$$

where

$$C(p) = \left\{\frac{1}{n^2}\left[\sum_{i=1}^{n-2}((n-i-1)p^4-(n-i-2)p^2-1)\right](1+(i-1)p)^2 + (p^2-1)p^2\right\}. \qquad (23)$$

Also, from Equation (20) and (21), we have

$$||z-x|| \le \frac{1}{n}\sum_{i=1}^{n-1}||x-T^ix|| \le \frac{1}{n}\sum_{i=1}^{n-1}(1+(i-1)p)||x-Tx|| = D(k)||x-Tx||,$$

where

$$0 < D(k) = \frac{1}{n}\sum_{i=1}^{n-1}(1+(i-1)p).$$

By Lemma 2.1 , if $C(p) < 1$, then Fix $T \neq \emptyset$. It follows from Equation (23) that if $n = 3$, then

$$2p^4 - p^2 - 10 < 0 \qquad (24)$$

Using Octave to obtain the optimal value of p in Equation (24), thus $\tilde{\gamma}_3^{\mathcal{H}} \ge 1.5811$. To evaluate $\tilde{\gamma}_4^{\mathcal{H}}$, we only need to solve $C(p) < 1$ for $n = 4$ as follows:

$$
\begin{aligned}
&\frac{1}{16}\left[\left(\sum_{i=1}^{2}(4-i-1)p^4-(4-i-2)p^2-1\right)(1+(i-1)p)^2 + p^2(p^2-1)\right] < 1 \\
&(2p^2-p^2-1) + (p^4-1)(1+p)^2 + (p^2-1)p^2 < 16 \\
&3p^4 - 2p^2 + (p^4-1)(1+p)^2 - 17 < 0. \qquad (25)
\end{aligned}
$$

Using Octave to obtain the optimal value of p in Equation (25), thus $\tilde{\gamma}_4^{\mathcal{H}} \ge 1.3251$.

From Equation (22), we have

$$
\begin{aligned}
||z-Tz||^2 &\le \frac{1}{n^2}\left[\sum_{i=1}^{n-1}((n-i-1)p^4-(n-i-2)p^2-1)\right]||x-T^ix||^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n-1}S_i||x-T^ix||^2,
\end{aligned}
$$

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where $S_i = (n-i-1)p^4 - (n-i-2)p^2 - 1$.

We consider the general cases for both even and odd values of n as follows:

1. Suppose n is odd, say $n = 2r+1$. Using the fact that T is uniformly p-lipschitzian and Equation (20), we have

$$\begin{aligned}
\sum_{i=1}^{n-1} S_i||x - T^i x||^2 &= \sum_{i=1}^{2r} S_i\|x - T^i x\|^2 \\
&= S_1\|x - Tx\|^2 + S_2\|x - T^2x\|^2 + ... \\
&\quad + S_r\|x - T^r x\|^2 + S_{r+1}\|x - T^{r+1}x\|^2 + ... \\
&\quad + S_{2r-1}\|x - T^{2r-1}x\|^2 + S_{2r}\|x - T^{2r}x\|^2 \\
&= S_1\|x - Tx\|^2 + S_2\|x - T^2x\|^2 + ... \\
&\quad + S_r\|x - T^r x\|^2 + S_{r+1}\|T^{2r+1}x - T^{r+1}x\|^2 + ... \\
&\quad + S_{2r-1}\|T^{2r+1}x - T^{2r-1}x\|^2 + S_{2r}\|T^{2r+1}x - T^{2r}x\|^2 \\
&\leq S_1\|x - Tx\|^2 + S_2\|x - T^2x\|^2 + ... \\
&\quad + S_r\|x - T^r x\|^2 + S_{r+1}p^2\|x - T^r x\|^2 + ... \\
&\quad + S_{2r-1}p^2\|x - T^2x\|^2 + S_{2r}p^2\|x - Tx\|^2 \\
&= (S_1 + S_{2r}p^2)\|x - Tx\|^2 + (S_2 + S_{2r-1}p^2)\|x - T^2x\|^2 + ... \\
&\quad + (S_r + S_{r+1}p^2)\|x - T^r x\|^2 \\
&= \sum_{i=1}^{r}(S_i + p^2 S_{2r-i+1})\|x - T^i x\|^2 \\
&= \sum_{i=1}^{r} Q(i, 2r+1)\|x - T^i x\|^2 \\
&\leq \sum_{i=1}^{r} Q(i, 2r+1)(1 + (i-1)p)^2\|x - Tx\|^2 \\
&= n^2 F_1||x - Tx||^2,
\end{aligned}$$

where

$$\begin{aligned}
S_i &= (n-i-1)p^4 - (n-i-2)p^2 - 1 \\
&= (2r+1-i-1)p^4 - (2r+1-i-2)p^2 - 1 \\
&= (2r-i)p^4 - (2r-i-1)p^2 - 1 \\
S_{2r-i+1} &= (i-1)p^4 - (i-2)p^2 - 1 \\
Q(i, 2r+1) &= S_i + p^2 S_{2r-i+1} \\
&= (2r-i)p^4 - (2r-i-1)p^2 - 1 + p^2((i-1)p^4 - (i-2)p^2 - 1) \\
&= (i-1)p^6 + (2r-2i+2)p^4 - (2r-i)p^2 - 1 \\
&= (i-1)p^6 + ((2r+1) - 2i + 1)p^4 - ((2r+1) - i - 1)p^2 - 1.
\end{aligned}$$

Hence, $Q(i, r) = (i-1)p^6 + (r-2i+1)p^4 - (r-i-1)p^2 - 1$.

2. Suppose n is even, say $n = 2r+2$. Using the fact that T is uniformly p-lipschitzian and Equation (20), we have

$$
\begin{aligned}
\sum_{i=1}^{n-1} S_i||x-T^ix||^2 &= \sum_{i=1}^{2r+1} S_i\|x-T^ix\|^2 \\
&= S_1\|x-Tx\|^2+S_2\|x-T^2x\|^2+... \\
&\quad +S_r\|x-T^rx\|^2+S_{r+1}\|x-T^{r+1}x\|^2+S_{r+2}||x-T^{r+2}x||+... \\
&\quad +S_{2r}\|x-T^{2r}x\|^2+S_{2r+1}\|x-T^{2r+1}x\|^2 \\
&= S_1\|x-Tx\|^2+S_2\|x-T^2x\|^2+... \\
&\quad +S_r\|x-T^rx\|^2+S_{r+2}\|T^{2r+2}x-T^{r+2}x\|^2+... \\
&\quad +S_{2r}\|T^{2r+2}x-T^{2r}x\|^2+S_{2r+1}\|T^{2r+2}x-T^{2r+1}x\|^2 \\
&\quad +S_{r+1}\|x-T^{r+1}x\|^2 \\
&\leq S_1\|x-Tx\|^2+S_2\|x-T^2x\|^2+... \\
&\quad +S_r\|x-T^rx\|^2+S_{r+2}p^2\|x-T^rx\|^2+... \\
&\quad +S_{2r}p^2\|x-T^2x\|^2+S_{2r+1}p^2\|x-Tx\|^2+S_{r+1}\|x-T^{r+1}x\|^2 \\
&= (S_1+S_{2r+1}p^2)\|x-Tx\|^2+(S_2+S_{2r}p^2)\|x-T^2x\|^2+... \\
&\quad +S_{r+1}\|x-T^{r+1}x\|^2 \\
&= \sum_{i=1}^{r}(S_i+p^2S_{2r-i+2})\|x-T^ix\|^2+S_{r+1}\|x-T^{r+1}x\|^2 \\
&= \sum_{i=1}^{r}Q(i,2r+2)\|x-T^ix\|^2+S_{r+1}\|x-T^{r+1}x\|^2 \\
&\leq \left[\sum_{i=1}^{r}Q(i,2r+2)(1+(i-1)p)^2+G(r)(1+rp)^2\right]\|x-Tx\|^2 \\
&= n^2F_2||x-Tx||^2,
\end{aligned}
$$

where

$$
\begin{aligned}
G(r) &= S_{r+1} \\
&= (2r+2-(r+1)-1)p^4-((2r+2)-(r+1)-2)p^2-1 \\
&= rp^4-(r-1)p^2-1.
\end{aligned}
$$

It follows from the two cases above that if $1 \leq i \leq r$ and

$$
\begin{aligned}
Q(i,r) &= (i-1)p^6+(r-2i+1)p^4-(r-i-1)p^2-1, \\
G(r) &= rp^4-(r-1)p^2-1, \\
F_1 &= \frac{1}{n^2}\sum_{i=1}^{r}Q(i,2r+1)(1+(i-1)p)^2, \\
F_2 &= \frac{1}{n^2}\left[\sum_{i=1}^{r}Q(i,2r+2)(1+(i-1)p)^2+G(r)(1+rp)^2\right],
\end{aligned}
$$

then

$$
||z-Tz||^2 \leq \begin{cases} F_1\|x-Tx\|^2 & \text{for } n=2r+1, \\ F_2\|x-Tx\|^2 & \text{for } n=2r+2. \end{cases} \tag{26}
$$

Thus, if n is odd and $F_1<1$ or if n is even and $F_2<1$, then $\mathrm{Fix}\,T \neq \emptyset$ and $\mathrm{Fix}\,T$ is a retract of $\mathbb{X}$.

To evaluate $\tilde{\gamma}_5^{\mathcal{H}}$, we only need to solve $F_1<1$ as follows:

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$$
\begin{aligned}
&Q(i,5) = (i-1)p^4 + (6-2i)p^4 - (4-i)p^2 - 1 \text{ and}\\
&F_1 = \frac{1}{25}\sum_{i=1}^{2} Q(i,5)(1+(i-1)p)^2 < 1\\
&\sum_{i=1}^{2}[(i-1)p^6 + (6-2i)p^4 - (4-i)p^2 - 1](1+(i-1)p)^2 < 25\\
&4p^4 - 3p^2 - 1 + (p^6 + 2p^4 - 2p^2 - 1)(1+p)^2 < 25\\
&4p^4 - 3p^2 + (p^6 + 2p^4 - 2p^2 - 1)(1+p)^2 - 26 < 0. \qquad (27)
\end{aligned}
$$

Using Octave to obtain the optimal value of p in Equation (27), we obtain $\tilde{\gamma}_5^{\mathcal{H}} \geq 1.2380$.

To evaluate $\tilde{\gamma}_6^{\mathcal{H}}$, we only need to solve $F_2 < 1$ as follows:

$$
\begin{aligned}
&Q(i,6) = (i-1)p^4 + (7-2i)p^4 - (5-i)p^2 - 1\\
&G(2) = 2p^4 - p^2 - 1\\
&F_2 = \frac{1}{36}\left[\sum_{i=1}^{2} Q(i,6)(1+(i-1)p)^2 + G(2)(1+2p)^2\right] < 1\\
&\sum_{i=1}^{2}[(i-1)p^4 + (7-2i)p^4 - (5-i)p^2 - 1](1+(i-1)p)^2 + (2p^4 - p^2 - 1)(1+2p)^2 < 36\\
&5p^4 - 4p^2 - 1 + (p^6 + 2p^4 - 2p^2 - 1)(1+p)^2 + (2p^4 - p^2 - 1)(1+2p)^2 < 36\\
&5p^4 - 4p^2 + (p^6 + 2p^4 - 2p^2 - 1)(1+p)^2 + (2p^4 - p^2 - 1)(1+2p)^2 - 37 < 0. \qquad (28)
\end{aligned}
$$

Using Octave to obtain the optimal value of p in Equation (28), we obtain $\tilde{\gamma}_6^{\mathcal{H}} \geq 1.1808$. □

Corollary 3.2.2. Let $\mathcal{H}$ be a Hilbert space and $b_i = \frac{1}{n}$, then $\tilde{\gamma}_3^{\mathcal{H}} \geq 1.5811$, $\tilde{\gamma}_4^{\mathcal{H}} \geq 1.3251$, $\tilde{\gamma}_5^{\mathcal{H}} \geq 1.2380$ and $\tilde{\gamma}_6^{\mathcal{H}} \geq 1.1808$.

4 Applications

Fixed point is one of the most useful tools in the study of nonlinear, algebraic, integral and differential equations. Many physical world problems can be modelled into fixed point problems.

Example 4.1. Consider the nonlinear algebraic equation $x^3 - x - 1 = 0$. There are several ways of putting the equation in the form $Tx = x$. The mapping T defined by $Tx = (x+1)^{\frac{1}{3}}$ is a contraction. Using the Mean Value Theorem, we obtain

$$
\begin{aligned}
|Tx - Ty| &= |(x+1)^{\frac{1}{3}} - (y+1)^{\frac{1}{3}}|\\
&= \frac{2^{\frac{1}{3}}}{6}|x-y|.
\end{aligned}
$$

Example 4.2. Let $\mathcal{N}$ be the 2-dimensional sphere defined as

$$\mathcal{N} = \{(a,z) \in \mathbb{R} \times \mathbb{C} : a^2 + |z|^2 = 1\}.$$

The isometry $T\mathcal{N} \to \mathcal{N}$ defined by

$$T(a,z) = (a, e^{\frac{2}{3}\pi i}), (a,z) \in \mathcal{N},$$

is a periodic map of order 3 with two fixed point (0,1) and $(0,-1)$.

5 Summary

The paper give an estimates for n-periodic k-lipschitzian and uniformly p-lipschitzian in a Hilbert space, in order to ensure the existence of fixed points. We obtained these results using the generalization of parallelogram law and Banach contraction mapping principle.

The general results of $\gamma_n^{\mathcal{H}}$ and $\tilde{\gamma}_n^{\mathcal{H}}, n \geq 3$ was demonstrated using the convex combinations $b_i = \frac{1}{n}$ for straightforward calculations. We find out that the approximations for $\tilde{\gamma}_n^{\mathcal{H}}$ is greater than $\gamma_n^{\mathcal{H}}$, which means that there are problems that cannot be solved using $T \in \alpha(k)$ but have solutions in $T \in \mathcal{U}(p)$.

Garcia and Nathansky [4] obtained $\gamma_5^{\mathcal{H}} \geq 1.1986$ and $\gamma_6^{\mathcal{H}} \geq 1.15$, we improved upon these results as shown in Table 1. It has been shown that γ_n has different forms in different spaces. We obtained new estimations for $\tilde{\gamma}_n$, $n = 3, 4, 5, 6$ for uniformly p-lipschitzian mappings. Our new estimates are better than previous results by [3] as shown in Table 2.

Table 1: Estimations of γ_n^X and $\gamma_n^{\mathcal{H}}$ for $T \in \alpha(k)$

	Koter (2000)	New Result		Gornicki and Pupka (2005)
$\gamma_3^{\mathcal{H}}$	1.3666	1.4678	γ_3^X	1.3821
$\gamma_4^{\mathcal{H}}$	1.1962	1.2905	γ_4^X	1.2524
$\gamma_5^{\mathcal{H}}$	1.0849	**1.2010**	γ_5^X	1.1777
$\gamma_6^{\mathcal{H}}$	1.0228	**1.1524**	γ_6^X	1.1329

Table 2: Estimations of $\tilde{\gamma}_n^X$ and $\tilde{\gamma}_n^{\mathcal{H}}$ for $T \in \mathcal{U}(p)$

	Koter (2000)	New Result		Gornicki and Pupka (2005)
$\tilde{\gamma}_3^{\mathcal{H}}$	1.5447	**1.5811**	$\tilde{\gamma}_3^X$	1.4558
$\tilde{\gamma}_4^{\mathcal{H}}$	1.2418	**1.3251**	$\tilde{\gamma}_4^X$	1.2917
$\tilde{\gamma}_5^{\mathcal{H}}$	1.1429	**1.2380**	$\tilde{\gamma}_5^X$	1.2001
$\tilde{\gamma}_6^{\mathcal{H}}$	1.0277	**1.1808**	$\tilde{\gamma}_6^X$	1.1482

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